AIAA JOURNAL
Vol. 41, No. 10, October 2003

Uncertainty Quantification of Structural Response Using Evidence Theory

Ha-Rok Bae* and Ramana V. Grandhi†
Wright State University, Dayton, Ohio 45435
and
Robert A. Canfield‡
Air Force Institute of Technology, Wright–Patterson Air Force Base, Ohio 45433

Over the past decade, classical probabilistic analysis has been a popular approach among the uncertainty quantification methods. As the complexity and performance requirements of a structural system are increased, the quantification of uncertainty becomes more complicated, and various forms of uncertainties should be taken into consideration. Because of the need to characterize the distribution of probability, classical probability theory may not be suitable for a large complex system such as an aircraft, in that our information is never complete because of lack of knowledge and statistical data. Evidence theory, also known as Dempster–Shafer theory, is proposed to handle the epistemic uncertainty that stems from lack of knowledge about a structural system. Evidence theory provides us with a useful tool for aleatory (random) and epistemic (subjective) uncertainties. An intermediate complexity wing example is used to evaluate the relevance of evidence theory to an uncertainty quantification problem for the preliminary design of airframe structures. Also, methods for efficient calculations in large-scale problems are discussed.

Nomenclature

A	=	set with its elements, $\{a_1, a_2, \ldots, a_n\}$
$A \subseteq B$	=	A a subset of B and B a superset of A
$\neg A$	=	negation of A
C	=	joint proposition
m	=	basic belief assignment function
U	=	set of a function value within a defined frame of discernment
U_F	=	set of failure system responses
X or Ω	=	frame of discernment
X_F	=	set of uncertain input vectors for failure of limit state
$x \in A$	=	x an element of A

Introduction

UNCERTAINTIES are classified into two distinct types in the risk assessment community: aleatory and epistemic uncertainty. Aleatory uncertainty is also called irreducible or inherent uncertainty. Epistemic uncertainty is a subjective or reducible uncertainty that stems from a lack of knowledge and data. Formal theories introduced to handle uncertainty include classical probability theory, possibility theory, and evidence theory. The common issue among these theories is how to determine the degree to which uncertain events are likely to occur. A distinct difference among these theories is in assignment of degree of belief.[1,2] Both classical probability theory and evidence theory limit the total belief for all possible events to be unity. On the other hand, there is no such restriction in possibility theory because one may have perfect confidence for a certain event and may give a possibility of one through a possibility distribution.

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*Graduate Research Assistant, Department of Mechanical and Materials Engineering; hbae@cs.wright.edu.

†Distinguished Professor, Department of Mechanical and Materials Engineering; rgrandhi@cs.wright.edu. Associate Fellow AIAA.

‡Lt. Colonel and Associate Professor, Department of Aeronautics and Astronautics, robert.canfield@afit.edu. Associate Fellow AIAA.

Probability theory, as a popular approach in uncertainty quantification in engineering structural problems, has been developed mostly for aleatory uncertainty, which often can be well represented by a probability distribution. The most familiar technique is Monte Carlo simulation. It generates random values of aleatory uncertain variables in a target system from a given probability distribution. The model is simulated with these random values to estimate the probability of a limit-state function. Besides Monte Carlo simulation, there are several well-developed methods for reliability analysis using probability theory: first-order reliability method[3] second-order reliability method,[4] stochastic finite element method,[5] and so on. However, there may exist some epistemic uncertainties to which probability theory is not appropriate because epistemic uncertainties often can not be assigned to every single event in a way that satisfies the axioms of probability theory. Hence, many researchers prefer the possibility theory to the probability theory to model these epistemic uncertainties in a system. Fuzzy set theory, which is closely related to possibility theory, was introduced by Zadeh in 1965.[6] Since then, fuzzy set theory has been studied in many areas. Structural design problems with fuzzy parameters were investigated by researchers such as Wood et al.[7] and Penmetsa and Grandhi.[8] However, in a real engineering structural system, as the complexity of the structural system increases, the propagation of uncertainty becomes more complicated. Furthermore, both aleatory and epistemic uncertainties need to be considered together. For instance, in an aircraft design, sufficient data for the dimensions and material properties of some parts may exist with probability distributions. However, there may not exist sufficient and credible information for other issues, such as gust loads, control surface settings, operating conditions, and so forth. Shafer[9] developed Dempster's work and presented an evidence theory, also called Dempster–Shafer theory, which makes it possible to handle both kinds of uncertainties, aleatory and epistemic.

Evidence theory is a generalization of classical probability and possibility theories from the perspective of bodies of evidence and their measures, even though the methodologies for manipulation of evidence are totally different. Hence, evidence theory can handle not only epistemic uncertainty, but also aleatory uncertainty. It allows for pre-existing probability information to be used together with epistemic information to assess likelihood for a limit-state function.

Until now, most applications of evidence theory have been for system maintenance or artificial intelligence, such as radio

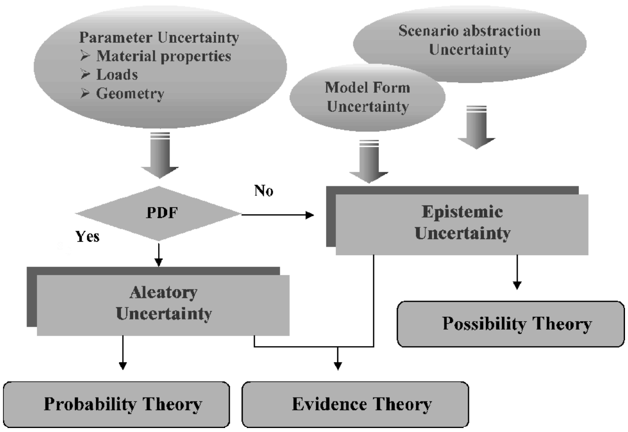

Fig. 1 Uncertainty quantification techniques.

communication systems, image processing, system management in nuclear industry, and decision making in design optimization problems.[10–12] Oberkampf and Helton[13] initially demonstrated evidence theory in quantifying uncertainty for a problem involving a closed-form equation. In this work, we attempt to apply evidence theory to a more practical problem definition. In evidence theory, a distribution of basic belief, which is similar to a probability distribution in probability theory, is constructed over all subsets of hypotheses. Hence, the information or hypotheses from each source are given as multiple intervals that might be nested in one another, or they might overlap. Dempster's combining rule will be used to fuse given intervals data from different independent sources. These are great advantages of evidence theory that are not addressed yet in uncertainty quantification for structural engineering problems. We present evidence theory methodology for an implicit analysis with a practical engineering system, the intermediate complexity wing (ICW) model. The strengths and weaknesses of evidence theory and improvements for solving large-scale uncertainty quantification problems are also discussed.

The uncertainty quantification strategy is depicted in Fig. 1 based on the characteristics of the information. Parameter uncertainties are basically aleatory uncertainties, but they can be epistemic uncertainties, when insufficient data are available to construct a probability distribution. Model form and scenario abstraction uncertainties, which are called epistemic uncertainties, can come from boundary conditions, different choices of solution approaches, unexpected failure modes, and so on, due to a lack of knowledge and information.

Evidence Theory

Evidence theory, also known as Dempster–Shafer theory, was originated by Dempster, and it was developed by Shafer[9] for both aleatory and epistemic uncertainties (see Ref. 14). Significant technical terms and definitions are discussed.

Frame of Discernment: X or Ω

Evidence theory starts by defining a frame of discernment that is a set of mutually exclusive elementary propositions. Any problem of likelihood takes some possible set as given. The given propositions might be nested in one another, or they might partially overlap. However, the finest possible subdivision of the set becomes the "elementary" proposition. The frame of discernment consists of all finite elementary propositions and may be viewed in the same way as a finite sample space in the probability theory.

In the case of a structural design problem, uncertainty can exist in the structural parameters of an analysis model as epistemic uncertainty. For the uncertain parameter, only interval information may be available, as shown in Fig. 2, and, in this case, the frame of discernment can be given as $X = \{x_1, x_2, x_3\}$, where, x_1, x_2, and x_3 are elementary propositions.

Various propositions can be expressed for negation, conjunction, and disjunction to elementary propositions. If we let 2^X denote the power set of X, then 2^X represents all of the possible distinct propositions. Hence, elementary propositions should be defined to reflect all of the available evidence within the power set of X, 2^X. The power set of $X = \{x_1, x_2, x_3\}$ is given as

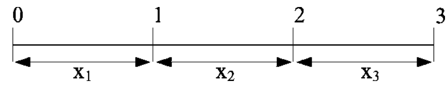

Fig. 2 Frame of discernment with elementary intervals.

$$2^X = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_3\}, X\}$$

Proposition $\{x_1, x_2\}$ in the set of 2^X means that one and only one of the two propositions is true, but we do not know which one is true. Because elementary propositions are selected to be mutually exclusive of each other, the true value of an uncertain parameter is assumed not to be located in both of the elementary propositions.

Basic Belief Assignment

In evidence theory, the basic propagation of information is through basic belief assignment (BBA). BBA expresses our degree of belief in a proposition. It is determined by various forms of information, sources, experimental methods, quantity and quality of information, and so forth. BBA is assigned by the use of a mapping function of BBA, m, to express our belief to a proposition with a number in the unit interval [0, 1]

$$m: 2^X \to [0, 1]$$

The number $m(A)$ represents only the portion of total belief assigned exactly to proposition A. The total belief will be obtained by consideration of belief and plausibility functions that will be discussed later. The measure m, BBA function, must satisfy the following three axioms:

$$m(A) \geq 0 \qquad \text{for any} \qquad A \in 2^X$$

$$m(\emptyset) = 0$$

$$\sum_{A \in 2^X} m(A) = 1$$

We do not assign any degree of belief to the empty proposition $\emptyset$, that is, we ignore the possibility for an uncertain parameter to be located outside of the frame of discernment in evidence theory. Though these three axioms of evidence theory look similar to those of probability theory, the axioms for the BBA functions are less restrictive than those for probability measure. In probability theory, the probability mass function p is defined only for an elementary, single proposition. On the other hand, in evidence theory, the given evidence may not exactly correspond to a defined elementary proposition. For instance, when a frame of discernment is given as $X = \{x_1, x_2, x_3\}$. Evidence may not be available for all of the single elementary propositions $\{x_1\}$, $\{x_2\}$, and $\{x_3\}$, but there may exist evidence for proposition $\{x_1, x_2\}$ that cannot be divided for two propositions $\{x_1\}$ and $\{x_2\}$ without employing additional assumptions.

Nevertheless, with the BBA mapping function, it is possible to assign BBA to any suitable proposition. Hence, the evidence for event $\{x_1, x_2\}$ can be used to assign the degree of belief, BBA, to the proposition $\{x_1, x_2\}$ directly, without being split into two propositions $\{x_1\}$ and $\{x_2\}$ individually. With the given evidence, it may not be possible to assign BBAs to all of the set 2^X, hence, BBA structure in evidence theory is more natural and intuitive way to express one's degree of belief in a situation where only partial evidence is available. In Fig. 3, the evidences for $\{x_1\}$, $\{x_2\}$, $\{x_1, x_2\}$, and X are available so that only BBAs for those propositions are defined. This can be elaborated with the following BBA structure:

$$m(\{x_1\}) = 0.5, \qquad m(\{x_2\}) = 0.3$$

$$m(\{x_1, x_2\}) = 0.1, \qquad m(X) = 0.1$$

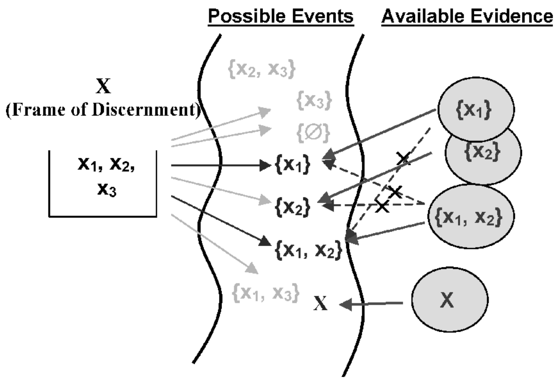

Fig. 3 Constructing BBA structure in evidence theory.

where $m(\{x_1\})$ means that we are sure with a 0.5 degree of basic belief from the evidence that x_1 is true. It is assumed that the given evidence is used only to define BBA for the proposition $\{x_1\}$, that is, the evidence does not imply $m(\{x_2\})+m(\{x_3\})$ or $m(\{x_2, x_3\})$. Hence, we can not give any BBA to $m(\{x_2\})$, $m(\{x_3\})$, or $m(\{x_2, x_3\})$ with the evidence for $\{x_1\}$. Based on this example, the following contrasts between evidence theory and probability theory are summarized:

1) In probability theory, additivity is one of the axioms $[p(a)+p(b)=p(a\cup b)]$. On the other hand, in evidence theory, it is not necessarily true. For instance, $m(\{x_1\})+m(\{x_2\})$ is not the same as $m(\{x_1, x_2\})$ because there is uncertainty in the given information. Moreover, the BBA for proposition $\{x_1, x_2\}$ is not obtained by adding up $m(\{x_1\})$ and $m(\{x_2\})$. However, it is obtained from the evidence for $\{x_1, x_2\}$, and the evidence might be independent of $m(\{x_1\})$ and $m(\{x_2\})$.

2) In probability theory, the probability for x_1, $p(x_1)$, should always be smaller than the probability for $x_1\cup x_2$, $p(\{x_1, x_2\})$. In evidence theory, evidence for $\{x_1\}$ is not transmitted to $\{x_1, x_2\}$, and, likewise, evidence for $\{x_1, x_2\}$ does not affect its subsets $\{x_1\}$ and $\{x_2\}$. Hence, it is possible that $m(\{x_1\})\geq m(\{x_1, x_2\})$ even though $\{x_1\}$ is a subset of $\{x_1, x_2\}$. Moreover, in evidence theory, we can not determine any distribution of BBA of proposition $\{x_1, x_2\}$ to its subsets.

3) It is not required that $m(X)=1$, but it is required that $m(X)\leq 1$. In probability theory, $p(\emptyset)=0$ implies that $p(X)=1$. However, in evidence theory, this implication is not accepted. BBA can be assigned only with supporting evidence or information.

In summary, BBA is not probability, but it is just a belief in a particular proposition irrespective of other propositions. The BBA structure gives the flexibility to express belief for possible propositions with partial and insufficient evidence and also avoids our making excessive or baseless assumptions in assigning our belief to propositions.

Dempster's Rule of Combining

Two BBA structures, m_1 and m_2, given by two different evidence sources, can be fused by Demspster's rule of combining to make a new BBA structure, as shown in Eq. (1),

$$m(A)=\frac{\sum_{C_i\cap C_j=A} m_1(C_i)m_2(C_j)}{1-\sum_{C_i\cap C_j=\emptyset} m_1(C_i)m_2(C_j)},\qquad A\neq\emptyset \tag{1}$$

where C_i and C_j denote propositions from each of the sources. In Eq. (1), the denominator can be viewed as a contradiction or conflict among the information given by independent knowledge sources. Even when some conflicts are found among the information, Dempster's rule disregards every contradiction by normalizing with the complementary degree of contradiction because it is designed to use consistent opinions from different sources as much as possible. However, this normalization can cause a counterintuitive and numerically unstable combining of information when the given information from independent sources has a lot of contradictions or conflicts.[15] In other words, Dempster's rule can be appropriate to a situation where there is some consistency and sufficient agreement among the opinions of different sources. On the other hand, Yager[16] has proposed an alternative rule of combination in which all contradiction is attributed to total ignorance. In this paper, we assume that there is enough consistency among given sources to use Dempster's rule of combining.

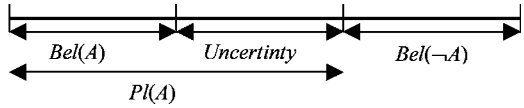

Fig. 4 Belief (bel) and plausibility (pl).

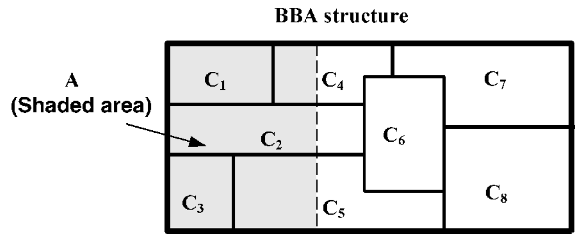

Fig. 5 Bel and pl in a given BBA structure.

Belief and Plausibility Functions

Because of a lack of information, it is more reasonable to present bounds for the result of uncertainty quantification, as opposed to a single value of probability. Our total degree of belief in a proposition A is expressed within a bound [bel(A), pl(A)], which lies in the unit interval [0, 1], as shown in Fig. 4, where bel( ) and pl( ) are given as belief function

$$\mathrm{bel}(A)=\sum_{C_i\subset A} m(C_i)$$

and plausibility function

$$\mathrm{pl}(A)=\sum_{C_i\cap A\neq\emptyset} m(C_i)$$

Because of uncertainty, the degree of belief for the proposition A and the degree of belief for a negation of the proposition A do not have to sum up to unity. Bel(A) is obtained by summation of the BBAs for propositions that are included in the proposition A. With this viewpoint, bel(A) is our total degree of belief. We called $m(C_i)$ a portion of total belief in the proposition A in the preceding section. The degree of plausibility pl(A) is calculated by the addition of the BBAs of propositions whose intersection with the proposition A is not an empty set. That is, every proposition that allows for the proposition A to be included, at least partially, is considered to imply the plausibility of proposition A because BBA in a proposition is not divided in any way to its subsets. Again, bel(A) is obtained by addition of the BBAs of propositions that imply the proposition A, whereas pl(A) is plausibility calculated by adding the BBAs of propositions that imply or could imply the proposition A. In a sense, these two measurements consist of lower and upper probability bounds.

For instance, Fig. 5 represents a BBA structure where the proposition A is expressed in the shaded area. The belief function, bel(A) is obtained by addition of the BBAs for C_1 and C_3 that are totally included in the shaded area. On the other hand C_1, C_2, C_3, C_4, and C_5 are added up for pl(A) because those propositions are partially or totally implying the proposition A.

Uncertainty Quantification Methodology in a Structural Application

Problem Definition

The structural responses can be expressed as a vector $\boldsymbol{Y}$ that depends on an input vector $\boldsymbol{X}$, with a system model f:

$$Y=f(X) \tag{2}$$

It is assumed that the variables in this model are independent of each other and uncertainties exist only in system parameters. When only parametric uncertainties in a system model are considered, the uncertainties in responses are determined by the uncertainties in the input parameters. The parametric uncertainty is typically included in aleatory uncertainty due to its inherent and stochastic nature. In the case of incomplete and insufficient information of the input parameters, they also could be represented as aleatory [e.g., by crude fits to probability density function (PDF)] but not very accurate. Hence, the nature of parameter uncertainty in insufficient information situations is better characterized as epistemic.

BBA Structure in Engineering Applications

In this work, we consider the situation that multiple intervals for an uncertain parameter in an engineering structure system are given by information sources, instead of an approximated PDF. Each interval represents a proposition of the true value of an uncertain parameter, and BBAs from each expert are assigned to each interval with mapping function m, based on the available evidence. The intervals can be discontinuous and scattered, and they may even overlap, as in Fig. 6.

In Fig. 6, the two subscripts are the indications for uncertain parameter and interval, respectively. From the information given in Fig. 6, the frame of discernment for the uncertain parameter x_1 is defined as an interval [0, 1.0]. The BBA structure satisfies the three axioms of BBA structure. As mentioned before, because the evidence is not transmitted to other propositions, the BBA of x_{11} can be higher than that of x_{14}, which includes the interval proposition x_{11}. When a proposition like x_{15} in given information indicates the frame of discernment, the BBA of the proposition is also viewed as the degree of ignorance. This is because the proposition can be interpreted in such a way that the information source has no idea how to give specific interval propositions in the frame of discernment with available partial evidence. Note that the BBA for an interval proposition is not distributed over the interval with any distribution function. When enough discretized intervals are obtained from available evidence, the BBA structure will express an unidentified PDF approximately.

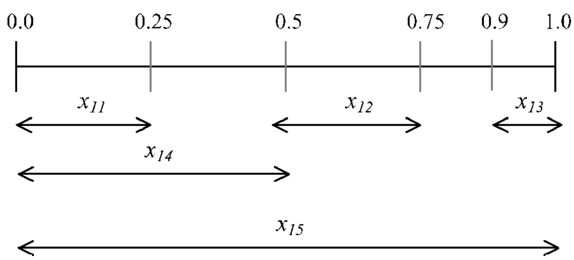

Fig. 6 Multiple interval information and BBA structure for an uncertain parameter x_1: x_{11} = [0, 0.25], x_{12} = [0.5, 0.75], x_{13} = [0.9, 1.0], x_{14} = [0, 0.5], x_{15} = [0, 1.0], $m(x_{11})$ = 0.4, $m(x_{12})$ = 0.15, $m(x_{13})$ = 0.1, $m(x_{14})$ = 0.25, and $m(x_{15})$ = 0.1.

Multiple sources for BBA structure, like two experts, are assumed. Dempster's rule of combining fuses interval information from independent sources, without refining the intervals. It is employed because there is no assumed distribution function of BBA within an interval. It is the basic concept in Dempster's rule of combining that the propositions, in agreement with other information sources, are given more credence. Those propositions are emphasized by the normalization of the complementary degree of contradiction in Dempster's rule of combining.

After the combined BBA structure for each uncertain parameter x_{ci} is obtained, the joint proposition, C, is constructed for the structural system model by use of the Cartesian product of each uncertain parameter. The joint BBA structure must follow the three axioms of BBA structure. For example, for only two uncertain parameters, the joint proposition is defined as

$$C = x_{c1} \times x_{c2} = \{c_k = [x_{c1m}, x_{c2n}] : x_{c1m} \in x_{c1}, x_{c2n} \in x_{c2}\} \tag{3}$$

The BBA for the joint proposition set is defined by

$$m_c(c_k) = m(x_{c1m})m(x_{c2n}) \tag{4}$$

Belief and Plausibility Function Evaluation

The two measurements of evidence theory, degree of plausibility and degree of belief, are obtained when the X_F set of uncertain input vectors and the U_F set of a failure system response are set, as in Eqs. (5) and (6). The failure occurrence of a target system response is defined with a limit-state value v:

$$U_F = \{y : y = f(\boldsymbol{x}) > v, \qquad \boldsymbol{x} = [x_1, x_2, \ldots, x_n] \in X\} \tag{5}$$

$$X_F = \{\boldsymbol{x} : y = f(\boldsymbol{x}) > v, \qquad \boldsymbol{x} = [x_1, x_2, \ldots, x_n] \in X\} \tag{6}$$

After determining the sets X_F and U_F, the belief and plausibility functions are evaluated by checking all propositions of the joint BBA structure

$$\text{bel}(U_F) = \sum_{c_k : c_k \subset X_F, c_k \in C} m_c(c_k) \tag{7}$$

$$\text{pl}(U_F) = \sum_{c_k : c_k \cap X_F \neq \emptyset, c_k \in C} m_c(c_k) \tag{8}$$

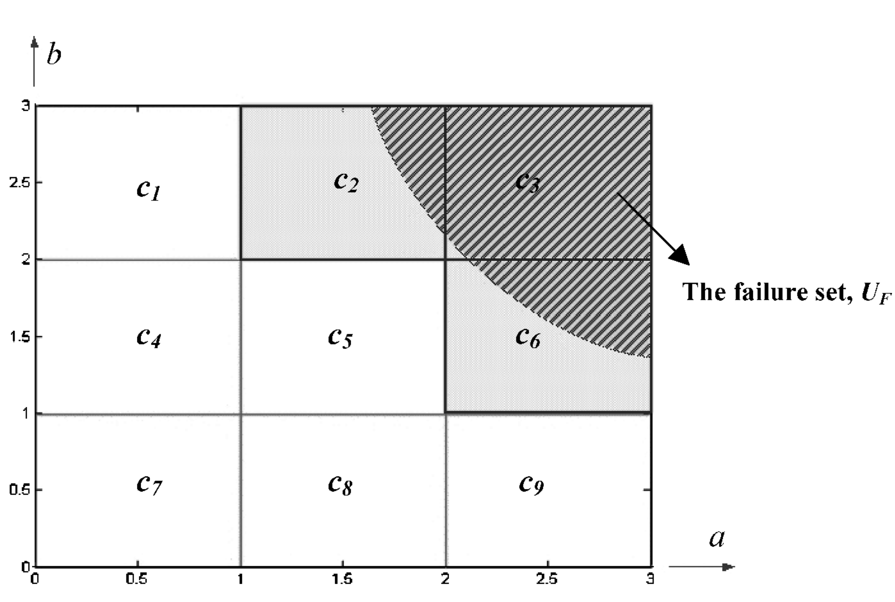

Fig. 7 Failure set U_F and joint BBA structure for two uncertain parameters, a and b.

Because the uncertain parameters in a joint proposition are continuous in an engineering application, it is required in the evaluation of belief and plausibility functions that the maximum and minimum responses over the joint proposition range be found:

$$[y_{\max}, y_{\min}] = \{\min[f(c_k)], \max[f(c_k)]\} \tag{9}$$

Then, by comparison of the range of system responses with the limit-state value v, the belief and plausibility functions are calculated. For instance, when joint propositions c_k border on each other in two-dimensional uncertain parameter space, as shown in Fig. 7, each joint proposition will be evaluated whether the response range of the joint proposition is included in the U_F set partially or entirely. Graphically, it is observed that joint propositions c_2, c_3, and c_6 are partially included in the U_F set, and the BBAs for those propositions will be added up for the degree of plausibility.

Several possible methods can be proposed to find the system response range for each joint proposition in engineering applications. They are the vertex method, sampling method, optimization method, approximation method, and so forth. When a system response is continuous and monotonic with respect to every uncertain parameter, the vertex method[17] can be used to find the system response range. However, when the limit-state function is expressed as a nonlinear function, as in many engineering applications, sampling or suboptimization techniques can be applied to find the maximum and minimum range values in each joint proposition. Those two techniques may require intolerable computational effort in a complex and large-scale system. Hence, to alleviate the computational requirement without sacrificing accuracy, a surrogate model can be introduced when available approximation methods can be used. To secure the accuracy of a surrogate model, the function space defined by the frame of discernment for a joint BBA structure can be divided into several subspaces, and surrogate models will be constructed over the subspaces. However, because it is our intention to introduce BBA structure and two measures of evidence theory to an uncertainty quantification problem of an engineering application, the simplest method, the vertex method, among the proposed methods, is used in the following example. Application of the vertex method is justified by the assumption that the target system model has linear variations in a small function space of each joint proposition with respect to every uncertain parameter.

The summary of the uncertainty quantification scheme using evidence theory is presented in Fig. 8. The following are the major steps in evaluating uncertainty with evidence theory.

The region marked by a dashed line in Fig. 8 has been implemented as a black box module for N number of uncertain parameters. After the information is combined for each parameter, the joint BBA structure is constructed under the assumption of independency of uncertain parameters. The joint BBA structure must follow the three axioms of BBA structure. The function evaluation spaces are determined when the joint BBA structure is constructed. The degrees of plausibility and belief are obtained when all of the joint propositions have been checked with the belief and plausibility functions.

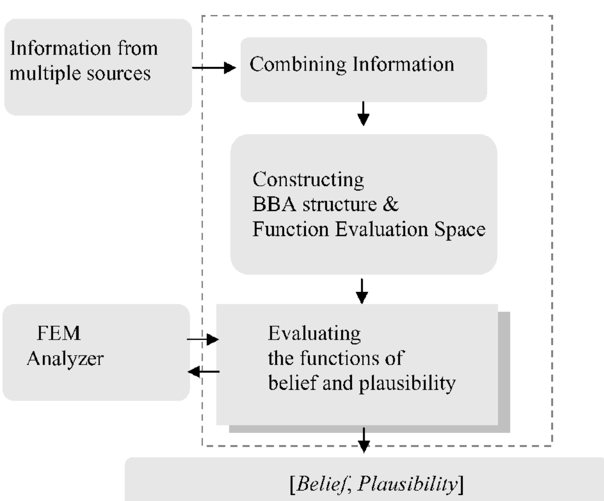

Fig. 8 Uncertainty quantification algorithm in evidence theory.

ICW Problem

Figure 9 shows the structural model of an ICW. There are 62 quadrilateral composite membrane elements for the upper and lower skins and 55 shear elements for the ribs and spars. Root chord nodes are constrained as supports. Static loads, which represent moments of aerodynamic lifting forces, are applied along the surface nodes, and the tip displacement at the identified point in Fig. 9 is considered a limit-state response function. It is assumed that there are four uncertain parameters: elastic modulus factor, load factor, and tip and root region of the wing skin thickness factors. The basic value for each parameter is fixed, and the real values are obtained by multiplication by the uncertain factors, for instance, the basic value of elastic modulus is 1.3×10^6 kg/cm^2. Physical linking is used for the skin thicknesses, and so there are two uncertain factors of wing thickness, the tip and root regions, for the skin thickness as shown in Fig. 9. We consider the situation where two experts (expert$_1$, expert$_2$) give their uncertain information for the four parameters with discontinuous and discrete intervals because the available data for the parameters are not enough to predict any variability. The interval information is considered to be the most appropriate way to express those variabilities, based on insufficient evidence.

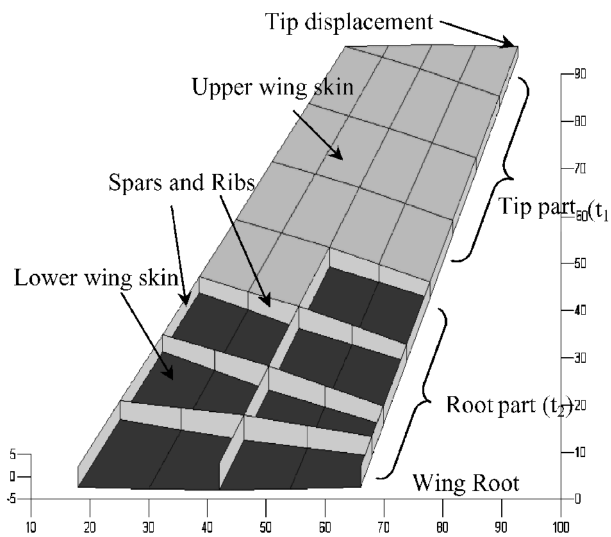

Fig. 9 ICW structure model.

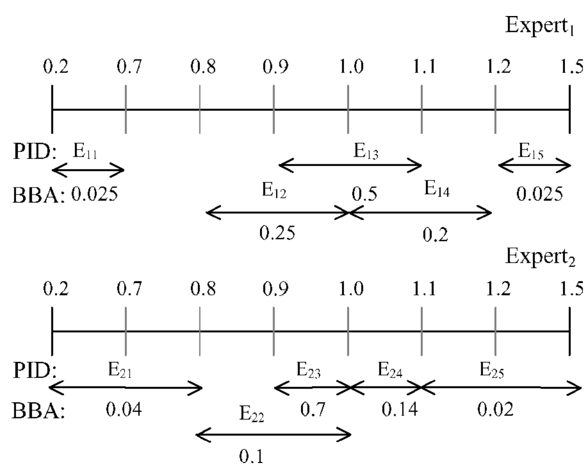

Fig. 10 Elastic modulus factor information.

Two equally credible experts are assumed to give their opinion with multiple intervals for each uncertain parameter. The interval information for elastic modulus and load are given in Figs. 10 and 11. Because of the lack of information, the interval information in evidence theory may not be continuous, and intervals can overlap. In Fig. 10, E_{11} indicates the first expert's first interval proposition for the E factor.

There is a discontinuous interval [0.7, 0.8] that is not covered by expert_1's opinion. That is, there is no evidence from expert_1 that supports the proposition that the elastic modulus factor exists in that interval. As mentioned before, even though the interval E_{22} includes the interval E_{23}, the BBA of E_{23} is higher than that of E_{22} because the evidence that is supporting the interval E_{23} is independent of the evidence supporting the interval E_{22}. This scheme allows us to express our opinion intuitively and realistically for given partial information, without making additional assumptions.

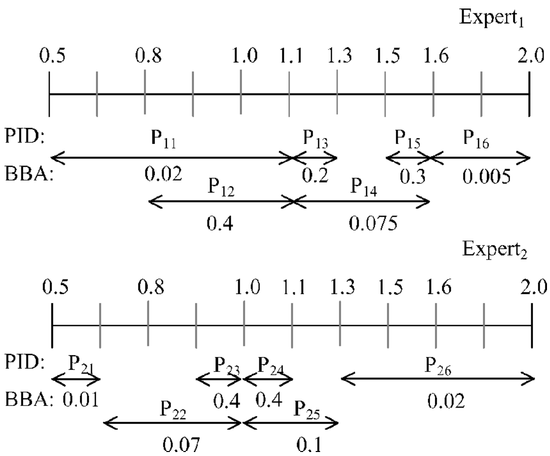

Fig. 11 Load factor information.

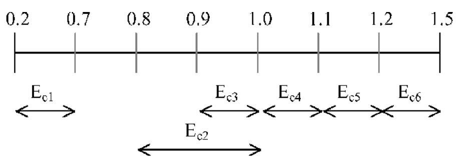

Fig. 12 Combined information for elastic modulus factor, BBAs: $E_{c1} = 0.0014$, $E_{c2} = 0.0355$, $E_{c3} = 0.8173$, $E_{c4} = 0.1393$, $E_{c5} = 0.0057$, and $E_{c6} = 0.007$.

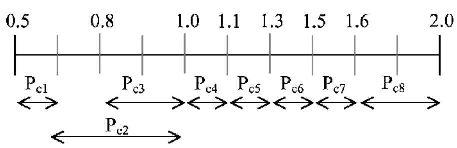

Fig. 13 Combined information for load factor, BBAs: $P_{c1} = 0.0005$, $P_{c2} = 0.0032$, $P_{c3} = 0.4427$, $P_{c4} = 0.4744$, $P_{c5} = 0.0621$, $P_{c6} = 0.0034$, $P_{c7} = 0.0136$, and $P_{c8} = 0.0002$.

The tip and root skin thickness factor information is shown in Tables 1 and 2.

The opinions from two different experts are consolidated with Dempster's rule of combining. For instance, the combined information for elastic modulus and load is given in Figs. 12 and 13.

The structural analyses were conducted with ASTROS[18] to obtain the tip displacements. Here, our goal is to obtain an assessment of the likelihood that the tip displacement exceeds the limit-state value of 0.5 in.:

$$U_F = \{\text{disp}_{\text{tip}} : \text{disp}_{\text{tip}} \geq 0.5\ \text{in.}\} \tag{10}$$

This goal is realized when the plausibility $\text{pl}(U_F)$ is obtained for the set of U_F with the joint BBA structure for uncertain parameters: elastic modulus, force, and thickness:

$$\text{pl}(U_F) = \sum_{\varepsilon_k : \varepsilon_k \cap X_F \neq \emptyset} m_c(c_k) \tag{11}$$

In this structural analysis problem, with four uncertain parameters, the vertex method requires 1800 function evaluations that are performed with ASTROS. As a result, belief is 0.0001 and plausibility is 0.0236 for exceeding the tip displacement limit state. This result

Table 1 Tip wing skin thickness factor, t_1

Expert_1		Expert_2	
Interval	BBA	Interval	BBA
[0.8, 1.0]	0.05	[0.8, 0.95]	0.1
[0.95, 1.05]	0.9	[0.95, 1.05]	0.85
[1.0, 1.2]	0.05	[1.05, 1.2]	0.05

Table 2 Root wing skin thickness factor, t_2

Expert_1		Expert_2			
Interval	BBA	Interval	BBA	Interval	BBA
[0.7, 0.9]	0.08	[0.7, 0.9]	0.03	[1.1, 1.3]	0.07
[0.9, 1.1]	0.82	[0.9, 1.0]	0.2	——	——
[1.1, 1.3]	0.1	[1.0, 1.1]	0.7	——	——

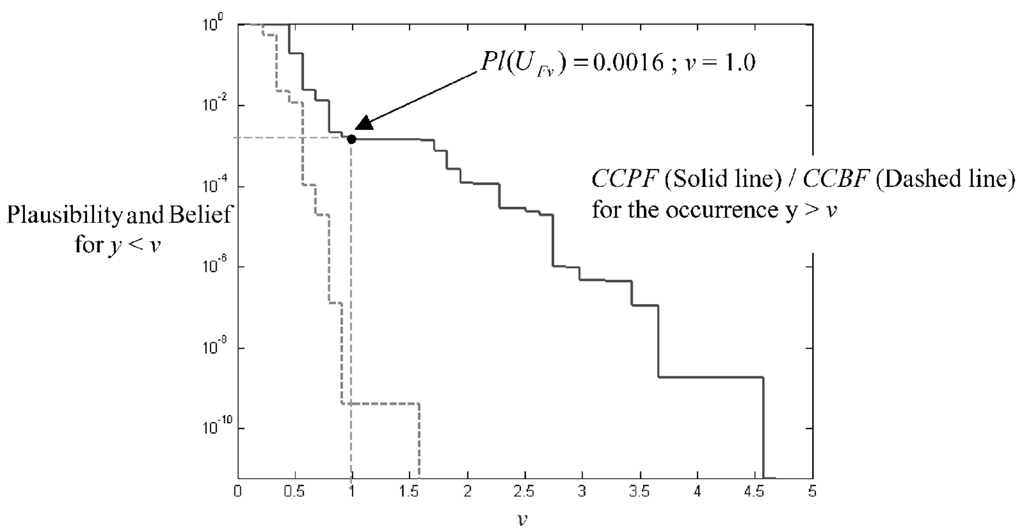

Fig. 14 CCPF and CCBF.

shows that degree of plausibility is 0.0236 for the failure of the wing structure regarding the tip displacement limit state, whereas there is at least 0.0001 belief for the failure. Belief and plausibility can be accepted as lower and upper bounds of an unspecified probability. Thus, a probability for U_F can be as low as 0.0001 and as high as 0.0236 with the given body of evidence. The complementary cumulative functions for plausibility (CCPF) and complementary cumulative functions for belief (CCBF) are defined with the functions pl and bel for the set U_{Fv}, with respect to the varying value of $v \in U$, where U and U_{Fv} are defined as

$$U = \{\mathrm{disp_{tip}} : \mathrm{disp_{tip}} = f(x),\ x = \{x_1, x_2, \ldots, x_n\} \in X\} \tag{12}$$

$$U_{Fv} = \{\mathrm{disp_{tip}} : \mathrm{disp_{tip}} \geq v,\ v \in U\} \tag{13}$$

The CCPF and CCBF are illustrated in Fig. 14. CCPF can be interpreted in the same way as cumulative distribution function in probability theory. For instance, when we want the plausibility for the occurrence $y > 1.0 (v = 1.0)$, the plausibility value 0.0016 is determined by the axis of plausibility of $y < v$, as indicated in Fig. 14.

The difference between plausibility and belief can be viewed as uncertainty, as shown in Fig. 4. Uncertainty reflects the lack of confidence in an analysis's result. Uncertainty varies with limit function value v in Fig. 14. When the available information is increased, uncertainty ultimately will be zero, and the three measures, plausibility, belief, and probability, will have the same value. In some cases, it is difficult to make a decision when uncertainty is too large. However, the bound $[\mathrm{bel}(U_{Fv}), \mathrm{pl}(U_{Fv})]$ is obtained based on given evidence and without any assumptions. Note that the two measures from evidence theory bracket the failure probability values that could result from any assumed probability distributions within the given interval information. Hence, we can say that the bound result from evidence theory is reasonably consistent with the given partial information. With this bound, we can obtain and apply the insight regarding the possible uncertainty in system response.

Summary

With the use of evidence theory, probability distributions are not needed, so that we can assess the reliability of a system with a small amount of information. In this work, the belief and plausibility measures are developed from what is typically referred to as a BBA structure that can be given as multiple intervals, with BBAs from different experts. This was the first attempt to develop the uncertainty quantification technique by the use of evidence theory in a structural system with multiple uncertain parameters. In the ICW example, partial information for uncertain parameters was expressed in a flexible and reasonable way through BBA structure. The bound result of the displacement limit-state function is found to be consistent and rigorous because no biased additional assumptions are employed on a distribution function of BBA. Evidence theory has been derived from classical probability theory, and so it is expected that preexisting information, such as a probability distributions, if available, can be incorporated. This will be the subject of future research for structural systems.

Acknowledgment

This project has been sponsored by the Air Force Office of Scientific Research through Grant F49620-00-1-0377.

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S. Saigal
Associate Editor